We want to solve $-\Delta u = 1$ on a domain $\Omega$ with Dirichlet boundary conditions. We first show that if we were to approximate the problem using piecewise linear finite elements on a mesh of size $h$, then

$$||\nabla(u - u_h)||^2 = ||\nabla u||^2 - ||\nabla u_h||^2$$

where $u_h$ is the finite element approximation on the mesh. Then show that the value of $||\nabla u_h||^2 = F \cdot A$ where $F$ is the load vector, and $A$ is the solution vector. Note that our variational form is

$$a(u, v) = \int_\Omega \nabla u \nabla v \qquad\qquad L(v) = \int_\Omega v \tag{1}$$

as $f = 1$.

1. Consider the following

$$\begin{aligned}\int_\Omega \big(\nabla(u - u_h)\big)^2 &= \int_\Omega (\nabla u - \nabla u_h)^2 \\ &= \|\nabla u\|^2 + \|\nabla u_h\|^2 - 2\int_\Omega \nabla u \nabla u_h\end{aligned}$$

   We want to show $-2\int_\Omega \nabla u \nabla u_h = -2\|\nabla u_h\|^2$. First, we integrate by parts and use $-\Delta u = 1$ to obtain $-2\int_\Omega \nabla u \nabla u_h = 2\int_\Omega \Delta u u_h = -2\int_\Omega u_h$. Now, use the variational form and that $u_h$ satisfies $a(u_h, u_h) = L(u_h)$.

2. The intuition here is that with the variational form, we have

$$a(u, u) = \int_\Omega (\nabla u)^2 = L(u) \implies \|\nabla u\|^2 = \int_\Omega fu.$$

   For our FE approximation version, we have

$$a(u_h, u_h) = \int_\Omega (\nabla u_h)^2 = L(u_h).$$

   Note that if we take a vector of all basis over $\Omega$

$$b = \begin{bmatrix}\phi_1 \\ \phi_2 \\ \vdots \\ \phi_n\end{bmatrix}$$

   and dot product it with $A$, we have $A \cdot b = u_h$ by definition. Furthermore, we know that

$$F = \begin{bmatrix}\int_{K_1} f\phi_1 \\ \int_{K_2} f\phi_2 \\ \vdots \\ \int_{K_n} f\phi_n\end{bmatrix} = \begin{bmatrix}\int_{K_1} \phi_1 \\ \int_{K_2} \phi_2 \\ \vdots \\ \int_{K_n} \phi_n\end{bmatrix}$$

Hence, we have that

$$F \cdot A = \sum \int_{K_n} \phi_n = L(u_h).$$

We know that $L(u_h) = a(u_h, u_h) = \|u_h\|^2$ and we are done.